

# Dynamic System Simplification using Rectified Logarithmic Pole Clustering Technique

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## ABSTRACT

The author suggested a rectified logarithmic pole clustering technique for simplification of the large-scale linear dynamic time-invariant systems. The denominator polynomial is determined by new pole clustering technique unlike logarithmic clustering technique already suggested by the authors<sup>[1]</sup>. The numerator of the reduced model is computed by matching few time-moments of the original system. The proposed method is easy to understand, efficient and capable to retain stability in the reduced models, if the original high order system is stable in nature. The viability of the proposed method is illustrated with the help of few examples taken from the literature.

**Keywords:** Order Reduction, Rectified Pole Clustering, Transfer function, Stability

The modelling of complex dynamic systems is one of the most important subjects in the science and engineering. Moreover, a model is often too complicated to be used in real problems, so approximations based on physical considerations or using mathematical approaches must be used to achieve simpler models than original one. The system simplification via reducing the order of its mathematical model in the form of transfer function plays a vital role in analysis and controller design. The various reasons for obtaining simplified models can be summarized as follows:

- ❖ To simplify the understanding of the system
- ❖ To reduce computational efforts in simulation problem
- ❖ To obtain simpler control law
- ❖ To design more efficient controller

A good number of research papers<sup>[1-6]</sup> are available in the literature to simplify the dynamic models in frequency and time domain. The several mathematical approaches have been suggested in frequency domain for synthesising the reduced order models of a large scale dynamic system. Few important algorithms are listed below:

- ❖ Pade approximations<sup>[7]</sup>
- ❖ Factor division algorithm<sup>[8]</sup>
- ❖ Stability equation method<sup>[9]</sup>
- ❖ Continued fraction expansion method<sup>[10]</sup>
- ❖ Differentiation method<sup>[11]</sup>
- ❖ Mihailov stability criterion<sup>[12]</sup>
- ❖ Routh approximation method<sup>[13]</sup>
- ❖ Dominant pole retention<sup>[14]</sup>
- ❖ Clustering technique<sup>[15]</sup>

Many algorithms for order reduction have been modified or improved by the researchers time to time. Few of them are listed below as:

- ❖ Modified Cauer continued fraction expansion<sup>[16]</sup>
- ❖ Simplified Routh approximation Method<sup>[17]</sup>
- ❖ Modified pole clustering technique<sup>[18]</sup>
- ❖ Logarithmic pole clustering<sup>[19]</sup>

The clustering technique initially suggested by J. Pal<sup>[15]</sup>, which was further improved to get more dominant pole cluster centres and this improvement was published as modified pole clustering technique<sup>[18]</sup>. An alternate approach for clustering of poles has been published before a couple of years, known as logarithmic pole clustering<sup>[19]</sup>. It is worthwhile to note that logarithmic of any value reduces its magnitude, is the main idea of this algorithm.

In this paper, the author suggested a rectification in the logarithmic pole clustering technique suggested by Jay<sup>[19]</sup>. The pole cluster centre should not be dependent on the 'order' of the simplified model and original system because the pole cluster centre is the dominant pole of that cluster only. The drawback of the algorithm<sup>[19]</sup> has been rectified in this research paper and suggested a correct and appropriate algorithm for model order reduction.

## Problem STATEMENT

Let the transfer of the high order dynamic system of the order ' $n$ ' be

$$G(s) = \frac{a_0 + a_1s + a_2s^2 + \dots + a_{n-1}s^{n-1}}{b_0 + b_1s + b_2s^2 + \dots + b_ns^n} \quad \dots(1)$$

Where  $a_i$  ( $0 \leq i \leq n-1$ ) and  $b_j$  ( $0 \leq j \leq n$ ) are known scalar constants.

Let the corresponding  $k^{th}$ -order simplified model is synthesized as,

$$R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{c_0 + c_1s + c_2s^2 + \dots + c_{k-1}s^{k-1}}{d_0 + d_1s + d_2s^2 + \dots + d_k s^k} \quad \dots(2)$$

Where  $c_i$  ( $0 \leq i \leq k-1$ ) and  $d_j$  ( $0 \leq j \leq k$ ) are unknown scalar constants.

The problem is to find simplified model  $R_k(s)$  from the original system  $G(s)$  in such a way that simplified model should match the important features of the original system.

## ALGORITHM FOR SYSTEM SIMPLIFICATION

An algorithm for system simplification via reducing the order of the original system consists of the following two steps:

**Step-1:** Determination of denominator polynomial of reduced order model using new pole clustering technique:

The logarithmic pole clustering initially suggested by the authors<sup>[19]</sup> in which, pole cluster centre depends on the parameters ' $k$ ' and ' $n$ ' which is not justified as well as practicable and hence a rectified logarithmic pole clustering technique is suggested in this paper which is independent on ' $k$ ' and ' $n$ '.

Let the  $i^{th}$ -cluster contains ' $r$ ' real poles as  $p_1, p_2, p_3, \dots, p_r$ , where,

$$|p_1| < |p_2| < |p_3| < \dots < |p_r|$$

The pole cluster centre ' $p_{ci}$ ' can be obtained from the above pole cluster by using the proposed algorithm as

$$p_{ci} = - \left\{ |p_1| + \left[ \log_{10} \left\{ 1 + \frac{|p_1| + |p_2| + \dots + |p_r|}{r^2} \right\} \div r^2 \right] \right\} \quad \dots(3)$$

Similarly, using equation (3) separately for real and imaginary parts of the complex poles, the complex pole cluster centre can be found.

Let the  $j^{th}$ -logarithmic pole cluster is obtained as,

$$\Lambda_{cj} = A_{cj} \pm jB_{cj} \text{ where } \Lambda^* = A_{cj} + jB_{cj} \text{ and } \dot{\Lambda} = A_{cj} - jB_{cj}$$

Similarly, pole cluster centre of positive poles cluster can be obtained using equation (3) with inverted sign. It is worthwhile to note that clusters of positive poles and negative poles must be made separately.

One of the following cases may be used to synthesize the denominator polynomial of the reduced simplified model:

**Case-1** if all cluster centres are real, then,

$$D_k(s) = (s - p_{c1})(s - p_{c2}) \dots (s - p_{ck}) \quad \dots(4)$$

**Case-2** if cluster centres are real and one pair of cluster centre is complex conjugate then,

$$D_k(s) = (s - p_{c1})(s - p_{c2}) \dots (s - p_{c(k-2)})(s - \Lambda_{c1}^*)(s - \dot{\Lambda}_{c1}) \quad \dots(5)$$

**Case-3** if all cluster centres are complex conjugates.

$$D_k(s) = (s - \Lambda_{c1}^*)(s - \dot{\Lambda}_{c1})(s - \Lambda_{c2}^*)(s - \dot{\Lambda}_{c2}) \dots (s - \Lambda_{ck/2}^*)(s - \dot{\Lambda}_{ck/2}) \quad \dots(6)$$

**Step-2:** Determination of numerator coefficients using time- moments matching.

The original system can be expanded about  $s = 0$  as follows:

$$G(s) = \frac{a_0 + a_1s + a_2s^2 + \dots + a_{n-1}s^{n-1}}{b_0 + b_1s + b_2s^2 + \dots + b_ns^n} = \sum_{i=0}^{\infty} T_i s^i \quad \dots(7)$$

The coefficients ' $T_i$ ' are known as time-moments of the original system. The method consists of  $k$  -number of equations for finding the numerator coefficients, is known as Pade approximation method<sup>[20]</sup>.

The time-moments  $T_i$  can also be calculated through the following equations

$$\left. \begin{aligned} T_0 &= \frac{a_0}{b_0} \\ T_i &= \frac{1}{b_0} \left[ a_i - \sum_{j=1}^i b_j T_{i-j} \right], i > 0 \\ a_i &= 0, \quad i > n-1 \end{aligned} \right\} \quad \dots(8)$$

The numerator coefficients ' $c_i$ ' can be calculated from following Pade equations as,

$$\left. \begin{aligned} c_0 &= d_0 T_0 \\ c_1 &= d_0 T_1 + d_1 T_0 \\ &\dots \\ &\dots \\ c_{k-1} &= d_0 T_{k-1} + d_1 T_{k-2} + \dots + d_{k-1} T_0 \end{aligned} \right\} \quad \dots(9)$$

## VALIDATION AND RESULT COMPARISON

Two large scale dynamic systems are taken from the literature to validate the proposed algorithm for

systems simplification. The original high order system is reduced to lower order models using the proposed algorithm. The time responses of the reduced simplified models are compared graphically with the original system. The error between the original and simplified models is computed with the help of MATLAB/simulink to check the effectiveness of the proposed method. The error is called as Integral Square Error (ISE)<sup>[21]</sup>, which is defined as,

$$ISE = \int_0^{\infty} [y(t) - y_k(t)]^2 dt \quad \dots(10)$$

Where  $y(t)$  and  $y_k(t)$  are step responses of the original and reduced simplified model respectively.

Example 1: consider an eight-order system from the literature<sup>[4]</sup>.

$$G(s) = \frac{N(s)}{D(s)}$$

$$N(s) = 18s^7 + 514s^6 + 5982s^5 + 36380s^4 + 122664s^3 + 222088s^2 + 185760s + 40320$$

$$D(s) = s^8 + 36s^7 + 546s^6 + 4536s^5 + 22449s^4 + 67284s^3 + 118124s^2 + 109584s + 40320$$

The poles are: -1, -2, -3, -4, -5, -6, -7, -8

Two clusters may be made as (-1,-2,-3,-4) and (-5,-6,-7,-8) for getting 2<sup>nd</sup> - order simplified model and using the Step-1, two cluster centres of respective clusters are obtained as,

$$p_{ci} = - \left[ -1 + \left[ \log_{10} \left\{ 1 + \frac{|-1| + |-2| + |-3| + |-4|}{4^2} \right\} \div 4^2 \right] \right] = -1.0131$$

Hence,  $D_2(s)$  can be obtained as,

$$D_2(s) = s^2 + 6.0393s + 5.0920$$

Few time-moments of the original system are computed from equation (8) and given as,

$$T_0 = 1 \quad T_1 = 1.889$$

From equations (9), numerator coefficients are computed as,

$$p_{ci} = - \left[ -5 + \left[ \log_{10} \left\{ 1 + \frac{|-5| + |-6| + |-7| + |-8|}{4^2} \right\} \div 4^2 \right] \right] = -5.0262$$

$$c_0 = d_0 T_0 = 5.0920 \times 1 = 5.0920$$

$$c_1 = d_0 T_1 + d_1 T_0 = 5.0920 \times 1.889 + 6.0393 \times 1 = 15.6581$$

So,  $N_2(s) = 15.6581s + 5.0920$

Finally, 2<sup>nd</sup> order simplified model is obtained as,

$$R_2(s) = \frac{15.6581s + 5.0920}{s^2 + 6.0393s + 5.0920}$$

The ISE of this simplified model is computed as 0.006934, which is very low as compared to many well known simplification methods and shown in Table 1.

**Table 1:** comparison of the proposed method

Simplification Method	Simplified Model	ISE
Proposed Method	$R_2(s) = \frac{15.6581s + 5.0920}{s^2 + 6.0393s + 5.0920}$	0.00693
Jay Singh <sup>[22]</sup>	$R_2(s) = \frac{15.6240s + 5.0748}{s^2 + 6.0306s + 5.0748}$	0.00683
G. Parmar <sup>[4]</sup>	$R_2(s) = \frac{24.1142s + 8}{s^2 + 9s + 8}$	0.0481
Mukherjee <i>et al.</i> <sup>[23]</sup>	$R_2(s) = \frac{11.3909s + 4.4357}{s^2 + 4.2122s + 4.4357}$	0.0569
Mittal <i>et al.</i> <sup>[24]</sup>	$R_2(s) = \frac{7.0908s + 1.9906}{s^2 + 3s + 2}$	0.2689
Hutton and Friedland <sup>[13]</sup>	$R_2(s) = \frac{1.98955s + 0.43184}{s^2 + 41.1736s + 0.43184}$	1.9170

Similarly, 3<sup>rd</sup> –order simplified model is obtained by considering the following pole cluster centres

Cluster-1: (-1, -2) and Cluster-2: (-3, -4, -5) and Cluster-3: (-6, -7, -8)

Respective pole clusters are obtained as,

$$p_{c1} = 1.0607$$

$$p_{c2} = 3.0408$$

$$p_{c3} = 6.0581$$

Hence, 3<sup>rd</sup> –order simplified model is obtained as,

$$R(s) = \frac{13.3635s^3 + 64.982s^2 + 19.5392s}{s^3 + 10.1596s^2 + 28.0725s + 19.5392}$$

The 2<sup>nd</sup> –order and 3<sup>rd</sup> –order simplified models are graphically comrade and shown in Fig. 1. The proposed method is also compared with the well- known reduction methods and comparative error is given in Table 1, from which it is clear that the proposed method is better than many methods available in the literature.

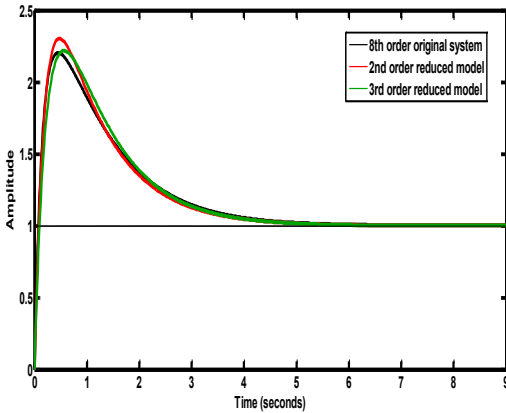


Fig. 1: Step response comparison for example-1

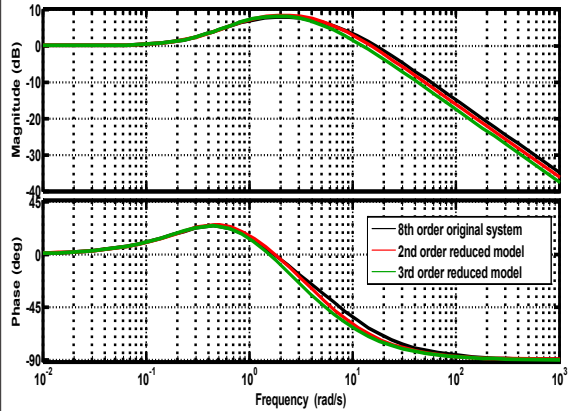


Fig. 2: Bode plot comparison for example-1

**Example-2:** Consider the 7<sup>th</sup>–order system of supersonic jet engine inlet investigated by Telescu by<sup>[25]</sup>, which is given in frequency domain as,

$$G(s) = \frac{N(s)}{D(s)}$$

$$N(s) = 25s^6 + 421.6s^5 + 10200s^4 + 95820s^3 + 870800s^2 + 3089000s + 10430000$$

$$D(s) = s^7 + 13.78s^6 + 612.7s^5 + 4730s^4 + 88120s^3 + 328100s^2 + 2894000s + 3020000$$

Poles of the system are:  $-1.1512, (-1.0733 \pm j7.0487), (-2.0554 \pm j11.9457), (-3.1857 \pm 18.4813)$

Few time-moments of the systems as  $T_0 = 3.4536, T_1 = -2.2866, T_2 = 2.1043$

Real and complex poles can be clustered separately and hence two pole clusters are obtained as,

$$p_{c1} = -1.1512$$

Complex pole cluster centre obtained as,

$$A_{c1} = -\left\{ \left| -1.0733 \right| + \left[ \log_{10} \left\{ 1 + \frac{\left| -1.0733 \right| + \left| -2.0554 \right| + \left| -3.1857 \right|}{3^2} \right\} + 3^2 \right] \right\} = -1.0989$$

$$B_{c1} = -\left\{ \left| -7.0487 \right| + \left[ \log_{10} \left\{ 1 + \frac{\left| -7.0487 \right| + \left| -11.9457 \right| + \left| -18.4813 \right|}{3^2} \right\} + 3^2 \right] \right\} = -7.1279$$

Hence,  $\Lambda_{cj} = -1.0989 \pm j7.1279$

Finally, 3<sup>rd</sup> –order reduced model is obtained as,

$$R_3(s) = \frac{12.857s^2 + 51.456s + 206.798}{s^3 + 3.349s^2 + 54.544s + 59.879}$$

The 3<sup>rd</sup> –order reduced model is obtained through proposed method and compared with time response of the original system shown in Fig. 2. The performance index ISE is computed and compared with other method<sup>[22]</sup> and given in Table 2.

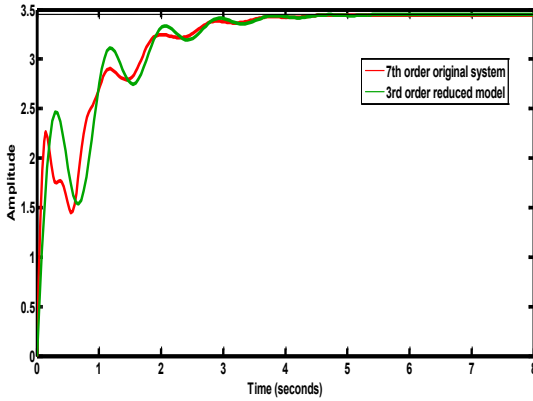


Fig. 3: Step response comparison for example-2

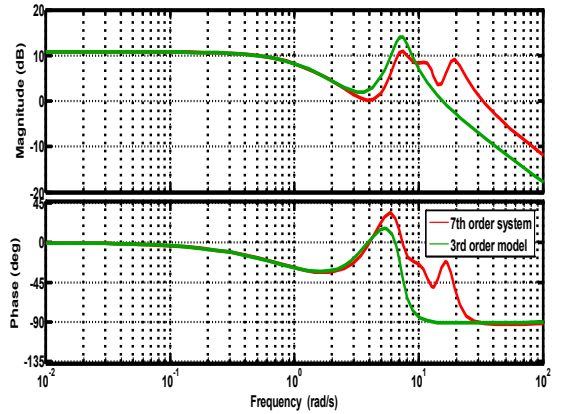


Fig. 4: Bode plot comparison for example 2

Table 2: Comparison with the other methods

Method	Simplified model	ISE
Proposed Method	$R_3(s) = \frac{12.857s^2 + 51.456s + 206.798}{s^3 + 3.349s^2 + 54.544s + 59.879}$	0.2344
Jay <sup>[22]</sup>	$R_3(s) = \frac{12.556s^2 + 50.99s + 203.8359}{s^3 + 3.3198s^2 + 53.836s + 59.1021}$	0.2457

From the above two examples, it is clear that suggested logarithmic clustering technique which has been rectified in this paper, is comparable in quality and able to retain stability features of the original system. Also, proposed method is better than many simplification methods suggested earlier in the literature.

## CONCLUSION

In this paper, author suggested a slight rectification for finding dominant pole from the pole clusters. It is worthwhile to state that the choice of formation of pole clusters depends upon the order of the reduced model but the dominant pole of any cluster does not depend upon the order of the reduced model. This concept has been incorporated and accordingly the logarithmic formula has been modified. Two numerical examples have been taken to illustrate and validate the suggested method for order reduction of the large-scale dynamic system. The time and frequency response comparisons are shown in Fig. 2 and 4 for the



both examples, from which it may be concluded that the response matching in time as well frequency domain is appreciable. Although, there is no significant improvement as compared to logarithmic pole clustering<sup>[19]</sup> but the correction made in the algorithm is technically correct and follow the main theme of pole clustering suggested by J. Pal<sup>[15]</sup>. It is absolutely clear that rectified logarithmic pole clustering is capable to retain the transient as well as steady-state properties of the original high order system. The proposed method also guarantees the retention of stability if the original high order system is stable.

The proposed algorithm is also applicable for linear and unstable single variable and multi-inputs and multi-outputs systems as well.

By using optimization techniques, the optimum numerator coefficients can be computed and hence better reduced models may be guaranteed.

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